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Explicit Error Bounds for Spline Interpolation on a Uniform Partition*

FRANÇOIS DUBEAU

Département de mathématiques et d'informatique, Faculté des Sciences, Université de Sherbrooke, 2500 boulevard Université, Sherbrooke, Québec, Canada, J1K 2R1

AND

JEAN SAVOIE

Département de mathématiques, Collège militaire royal de Saint-Jean, Richelain, Quebec, Canada, J0J 1R0

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This paper considers the optimality and the evaluation of the constants that appear in the expressions of error bounds for interpolating spline functions over a uniform mesh of the real line when the nodes are uniformly shifted. © 1995 Academic Press, Inc.

1. INTRODUCTION

A n-degree spline s defined over the uniform partition $\pi_h^a = a + \mathbb{Z}h$ of mesh size h of the real line $\mathbb{R}$ is a function $s \in C^{n-1}(\mathbb{R})$ such that s restricted to $[a + lh, a + (l+1)h]$ is an algebraic polynomial of degree at most n for any $l \in \mathbb{Z}$.

To simplify we use the notation $x_{l+t} = a + (l+t)h$. For a function f defined on $\mathbb{R}$ and $t \in \mathbb{R}$, $f_{l+t} = f(x_{l+t})$ and the shift operator is $Ef_t = f_{t+1}$.

Using the polynomials

$$p_n(t, z) = \sum_{i=0}^{n} Q_n(n+t-i)\, z^i \tag{1.1}$$

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for $t \in [0, 1]$ and $z \in \mathbb{R}$, where Q_n is the B-spline of degree n defined on π_1^0, ter Morsche [12], [13] obtained the following linear dependance relationships for a n-degree spline s defined on π_h^a

$$h^k p_n(v, E)\, s_{l+u}^{(k)} - p_{n-k}(u, E)(E-I)^k s_{l+v} = 0 \tag{1.2}$$

for any $l \in \mathbb{Z}$, u, $v \in [0, 1]$ and $k = 0, ..., n$.

A spline s is said to be the interpolating spline of f if, for a given $v \in [0, 1]$, we have $s_{l+v} = f_{l+v}$ for all $l \in \mathbb{Z}$.

A function f is said to be of polynomial growth on $\mathbb{R}$ if there exists an integer $v \geqslant 0$ such that $f(x) = O(|x|^v)$ for $|x| \to +\infty$.

A consequence of (1.2) and the properties of $p_n(t, z)$ is the existence and uniqueness of the n-degree interpolating spline s for any function f of polynomial growth when $p_n(t, -1) \neq 0$ (see [7], [8], [11], [12], and [6]).

Let us consider the following function spaces

$$L^1_{\text{loc}}(\mathbb{R}) = \left\{ f\colon \mathbb{R} \to \mathbb{R} \;\middle|\; \int_a^b |f(x)|\, dx < \infty, \text{ for all interval } [a, b] \subset \mathbb{R} \right\}$$

and

$$AC^{n+1}_{\text{loc}}(\mathbb{R}) = \left\{ f \in C^n(\mathbb{R}) \;\middle|\; \begin{array}{l} \text{(i)} \quad f^{(n+1)} \in L^1_{\text{loc}}(\mathbb{R}) \\ \\ \text{(ii)} \quad \text{For all } [a, b] \subset \mathbb{R},\, f^{(n)}(x)|_b^a = \int_a^b f^{(n+1)}(x)\, dx \end{array} \right\}.$$

For any $f \in AC^{n+1}_{\text{loc}}(\mathbb{R})$, using its Taylor expansion and the fact that (1.2) is satisfied for any polynomials of degree at most n, we obtain

$$\begin{aligned} h^k p_n(v, E)\, f_{l+u}^{(k)} &- p_{n-k}(u, E)(E-I)^k f_{l+v} \\ &= \frac{h^{n+1}}{n!} \int_0^{n+1} K_n^k(u, v, \theta)\, f_{l+\theta}^{(n+1)}\, d\theta \end{aligned} \tag{1.3}$$

for any $l \in \mathbb{Z}$, $u \in [0, 1]$, and $k = 0, ..., n$. This is nothing but a consequence of the Peano Kernel Theorem (see also [4] and [6]).

From (1.2) and (1.3) we obtain

$$h^k p_n(v, E)\, e_{l+u}^{(k)} = \frac{h^{n+1}}{n!} \int_0^{n+1} K_n^k(u, v, \theta)\, f_{l+\theta}^{(n+1)}\, d\theta \tag{1.4}$$

where $e = f - s$. Moreover, if $f^{(n+1)}$ is of polynomial growth, we have

$$e_{l+u}^{(k)} = \frac{h^{n+1-k}}{n!}\, p_n(v, E)^{-1} \int_0^{n+1} K_n^k(u, v, \theta)\, f_{l+\theta}^{(n+1)}\, d\theta. \tag{1.5}$$

Finally, since the norm of the operator $p_n(v, E)^{-1}$ on bounded sequences is upper bounded by $1/|p_n(v, -1)|$ (see [3], [5], and [12]), if $f^{(n+1)} \in L^\infty(\mathbb{R})$ we have

$$|e^{(k)}_{l+u}| \leqslant C^k_n(u, v)\, h^{n+1-k}\, \|f^{(n+1)}\|_\infty \tag{1.6}$$

and

$$\|e^{(k)}\|_\infty \leqslant C^k_n(v)\, h^{n+1-k}\, \|f^{(n+1)}\|_\infty \tag{1.7}$$

for $k = 0, ..., n$, where

$$C^k_n(u, v) = \frac{1}{|p_n(v, -1)|} \int_0^{n+1} \frac{|K^k_n(u, v, \theta)|}{n!}\, d\theta, \tag{1.8}$$

$$C^k_n(v) = \sup_{u \in [0, 1]} C^k_n(u, v). \tag{1.9}$$

In this paper we show that $C^0_n(u, v)$ and $C^0_n(v)$ are the best constants in (1.6) and (1.7). We also present explicit expressions for those constants. In the case $k > 0$ we present explicit expressions to bound the constants $C^k_n(u, v)$ and $C^k_n(v)$. These results are presented in sections 4 and 5. In section 2 we present some preliminaries and in section 3 we establish some useful properties of the kernels $K^k_n(u, v, \theta)$.

2. Preliminaries

The B-spline Q_n of degree n with knots $0, 1, ..., n+1$ can be defined by

$$Q_n(x) = \frac{\nabla^{n+1}(x)^n_+}{n!}$$

where ∇ is the backward difference operator, $(x)^n_+ = x^n \chi_{(0, +\infty)}(x)$ and χ_E is the characteristic function of the set E. It is also equivalent to the formula

$$Q_n(x) = \underbrace{\chi_{(0,1]} * \cdots * \chi_{(0,1]}}_{n+1}(x)$$

where $*$ denotes the convolution operator. Let us remark that

$$Q_n(x) = Q_{n-k} * Q_{k-1}(x)$$

for any $k = 1, ..., n$. Moreover, for any $f \in AC^k(\mathbb{R})$ we have

$$\nabla^k f(v) = Q_{k-1} * f^{(k)}(v). \tag{2.1}$$

The polynomials $p_n(t, z)$ defined by (1.1) for $t \in [0, 1]$ have the following properties (see [12] or [3]): $p_0(t, z) = \chi_{(0,1]}(t)$ and for $n > 0$:

$p_n(t, z)$ is a polynomial in z of degree n for each $t \in (0, 1]$ and $p_n(0, z)$ is of degree $n-1$;

$$p_n(t, -1) = 0 \quad \text{iff} \quad t = \tau_n = \begin{cases} \frac{1}{2} & \text{for } n \text{ odd}, \\ 0 (\text{or } 1) & \text{for } n \text{ even}; \end{cases} \tag{2.2}$$

$$\frac{\partial^k}{\partial t^k} p_n(t, z) = (z-1)^k \, p_{n-k}(t, z).$$

We can extend the definition of $p_n(t, z)$ for all $t \in \mathbb{R}$ by the formula

$$p_n(t, z) = \sum_{i=-\infty}^{\infty} Q_n(n + t - i) \, z^i.$$

It follows that

$$p_n(t+1, z) = z p_n(t, z). \tag{2.3}$$

Moreover $p_n(t, z)$ is a spline of degree n with respect to t and $p_n(t, 1) = 1$.

Finally we have the following useful property

$$p_n(t, z) = Q_{k-1}(t) * p_{n-k}(t, z) \, z^k \tag{2.4}$$

for $k = 1, ..., n$.

We will also use the Euler splines as defined in [10, p. 152]. The n-degree Euler spline defined on π_1^0, denoted by E_{n+1}, is such that

$$E_1(t) = (-1)^i \quad \text{for} \quad t \in (i, i+1] \text{ and } i \in \mathbb{Z},$$

and for $n > 1$

$$\frac{d}{dt} E_n(t) = 2E_{n-1}(t),$$

$$E_n(t+1) = -E_n(t),$$

$$E_n(1-t) = (-1)^{n+1} E_n(t),$$

$$\text{sign}(E_{n+2}(t)) = -\text{sign}(E_n(t)).$$

From the definition, it follows that

$$\max_{0 \leqslant t \leqslant 1} |E_{n+1}(t)| = |E_{n+1}(\tau_n^*)|$$

where

$$\tau_n^* = \begin{cases} \frac{1}{2} & \text{for } n \text{ even}, \\ 0(\text{or } 1) & \text{for } n \text{ odd}. \end{cases}$$

We also have $|\sin(\pi t)| \leqslant |E_1(t)| \leqslant 1$, and if we set for $n \geqslant 2$

$$g_n(t) = \begin{cases} \cos(\pi t) & \text{for } n \text{ even}, \\ \sin(\pi t) & \text{for } n \text{ odd}, \end{cases}$$

then

$$|g_n(t)| \left(\frac{2}{\pi}\right)^{n-1} \leqslant |E_n(t)| \leqslant |g_n(t)| \left(\frac{2}{\pi}\right)^{n-2}. \tag{2.5}$$

Finally, $p_n(t, -1) = (-1)^n E_{n+1}(t)$ and it follows that

$$\max_{0 \leqslant t \leqslant 1} |p_n(t, -1)| = |p_n(\tau_n^*, -1)| \geqslant \left(\frac{2}{\pi}\right)^n. \tag{2.6}$$

3. Analysis of the Peano Kernels

The Peano kernels in (1.3) are defined by

$$\frac{K_n^k(u, v, \theta)}{n!} = p_n(v, E) \frac{(u-\theta)_+^{n-k}}{(n-k)!} - p_{n-k}(u, E)(E-I)^k \frac{(v-\theta)_+^n}{n!} \tag{3.1}$$

for $k = 0, ..., n$. In the next five lemmas we present some useful properties of these kernels.

The first two lemmas are consequences of the consistency relations (1.2).

Lemma 1. *$K_n^k(u, v, \theta) = 0$ whenever $\theta \notin [\min\{u, v\}, n + \max\{u, v\}]$.*

Proof. Let us define the polynomial $g_\theta(x) = (x-\theta)^n/n!$. If $\theta < \min\{u, v\}$ then

$$\begin{aligned} K_n^k(u, v, \theta) &= p_n(v, E) \frac{(u-\theta)_+^{n-k}}{(n-k)!} - p_{n-k}(u, E)(E-I)^k \frac{(v-\theta)_+^n}{n!} \\ &= p_n(v, E) \frac{(u-\theta)^{n-k}}{(n-k)!} - p_{n-k}(u, E)(E-I)^k \frac{(v-\theta)^n}{n!} \\ &= p_n(v, E)\, g_\theta^{(k)}(u) - p_{n-k}(u, E)(E-I)^k g_\theta(v) = 0. \end{aligned}$$

For $\theta > n + \max\{u, v\}$, we obtain directly that $K_n^k(u, v, \theta) = 0$. ∎

LEMMA 2. *$K_n^k(u, v, i) = 0$ for any integer i.*

Proof. Let us define the function $h_\theta(x) = (x-\theta)_+^n/n!$. Whenever θ is an integer, the function $h_\theta(x)$ is a spline on π_1^0. Thus, by (1.2), we obtain

$$0 = p_n(v, E)\, h_\theta^{(k)}(u) - p_{n-k}(u, E)(E-I)^k\, h_\theta(v)$$
$$= \frac{K_n^k(u, v, \theta)}{n!}. \quad \blacksquare$$

LEMMA 3. *The kernel $K_n^0(u, v, \theta)$ has the following properties.*

(a) *$K_n^0(v, v, \theta) = 0$;*

(b) *$K_n^0(u, v, \theta)$ has no sign change for $\theta \in (i, i+1)$, and has simple zeros at $\theta = i$ for $i = 1, ..., n$;*

(c) *if $\theta \in [n, n + \max\{u, v\}]$ then sign $(K_n^0(u, v, \theta)) = sign(u - v)$.*

Proof. (a) Obvious.

(b) Let $u < v$. The kernel $K_n^0(u, v, \theta)$ is a n-degree spline with respect to θ defined on the partition $\{i+u, i+v \mid i = 0, ..., n\}$ and support $[u, n+v]$. For $\theta \in (u, v)$ we have $K_n^0(u, v, \theta) = -Q_n(n+v)(u-\theta)^n \neq 0$ and for $\theta \in (n+u, n+v)$ we have $K_n^0(u, v, \theta) = -Q_n(u)(n+v-\theta)^n \neq 0$. From [9, p. 155] the intervals $(-\infty, u)$ and $(n+v, \infty)$ are two zeros of multiplicity $n+1$. It follows from lemma 2 that the number of zeros of $K_n^0(u, v, \theta)$ is at least $3n+2$. But from [9, p. 160–161] the number of zeros of $K_n^0(u, v, \theta)$ is at most $3n+2$. Hence the result follows. A similar proof holds for $u > v$.

(c) If $u < v$ and $\theta \in (n+u, n+v)$ we have $K_n^0(u, v, \theta) = -Q_n(u)(n+v-\theta) < 0$, and if $u > v$ and $\theta \in (n+v, n+u)$ we have $K_n^0(u, v, \theta) = Q_n(v)(n+u-\theta) > 0$. The result follows. $\blacksquare$

The next lemma is a direct consequence of (2.3).

LEMMA 4. *For any $i \in \mathbb{Z}$ we have*

$$K_n^k(u+i, v, \theta) = K_n^k(u, v+i, \theta) = K_n^k(u, v, \theta - i). \quad \blacksquare$$

The next lemma relates $K_n^k(u, v, \theta)$ to $K_{n-k}^0(u, v, \theta)$ using the convolution operator.

LEMMA 5. *For $k = 1, ..., n$ we have*

$$\frac{K_n^k(u, v, \theta)}{n!} = Q_{k-1}(v) * \frac{K_{n-k}^0(u, v, \theta - k)}{(n-k)!}.$$

Proof. From (2.4) we have $p_n(v, E) = Q_{k-1}(v) * p_{n-k}(v, E)\, E^k$. Also $(E-I)^k = \nabla^k E^k = E^k \nabla^k$, and from (2.1), we have

$$\nabla^k \frac{(v-\theta)_+^n}{n!} = Q_{k-1}(v) * \frac{(v-\theta)_+^{n-k}}{(n-k)!}.$$

Finally, using the lemma 4 we obtain the result. ∎

4. The Case $k = 0$

This section contains the proof of the optimality of the constants $C_n^0(u, v)$ and $C_n^0(v)$ defined by (1.8) and (1.9).

Let us consider the following class of functions

$$\mathscr{C} = \{ f \in AC_{\text{loc}}^{n+1}(\mathbb{R}) \mid f^{(n+1)} \in L^\infty(\mathbb{R}) \}.$$

The results of this section are based on the following lemma.

Lemma 6. *Let $v \in [0, 1]$ such that $p_n(v, -1) \neq 0$. Then*

$$\int_0^{n+1} \frac{|K_n^0(u, v, \theta)|}{n!}\, d\theta$$
$$= \frac{\operatorname{sign}(u-v)}{2^{n+1}} \left(E_{n+1}(v)\, E_{n+2}(u) - E_{n+1}(u)\, E_{n+2}(v) \right) \qquad (4.1)$$

for any $u \in [0, 1]$.

Proof. Let $f(x) = (1/2^{n+1})\, E_{n+2}(x/h)$. Then $f^{(n+1)}(x) = (1/h^{n+1})\, E_1(x/h)$, $\|f^{(n+1)}\|_\infty = 1/h^{n+1}$, and $f \in \mathscr{C}$. Its n-degree interpolating spline s such that $f_{l+v} = s_{l+v}$ is $s(x) = (E_{n+2}(v)/2^{n+1} E_{n+1}(v))\, E_{n+1}(x/h)$. From (1.4) we have

$$p_n(v, E)\, e_{l+u} = \int_0^{n+1} \frac{K_n^0(u, v, \theta)}{n!} E_1(l+\theta)\, d\theta.$$

Using the definition of E_1 and the properties of the zeros of $K_n^0(u, v, \theta)$ given by the lemma 3, we have

$$\int_0^{n+1} \frac{K_n^0(u, v, \theta)}{n!} E_1(l+\theta)\, d\theta = (-1)^{l+n} \operatorname{sign}(u-v) \int_0^{n+1} \frac{|K_n^0(u, v, \theta)|}{n!}\, d\theta.$$

Moreover, we also have $p_n(v, E)\, e_{l+u} = p_n(v, -1)\, e_{l+u}$ because $e_{l+i+u} = (-1)^i e_{l+u}$. But

$$e_u = \frac{1}{2^{n+1}} \left(E_{n+2}(u) - E_{n+1}(u) \frac{E_{n+2}(v)}{E_{n+1}(v)} \right)$$

and $p_n(v, -1) = (-1)^n E_{n+1}(v)$. Hence the result follows. ∎

THEOREM 1. *Let $v \in [0, 1]$ be such that $p_n(v, -1) \neq 0$. If $f \in \mathscr{C}$, s is the n-degree interpolating spline of f such that $f_{l+v} = s_{l+v}$ $(l \in \mathbb{Z})$ and $e = f - s$, then*

$$C_n^0(u, v) = \max_{f \in \mathscr{C}} \frac{\sup_{l \in \mathbb{Z}} |e_{l+u}|}{h^{n+1} \|f^{(n+1)}\|_\infty}. \tag{4.2}$$

and

$$C_n^0(u, v) = \frac{1}{2^{n+1}} \left| E_{n+2}(u) - E_{n+1}(u) \frac{E_{n+2}(v)}{E_{n+1}(v)} \right|. \tag{4.3}$$

Moreover

$$C_n^0(v) = \max_{f \in \mathscr{C}} \frac{\|e\|_\infty}{h^{n+1} \|f^{(n+1)}\|_\infty} \tag{4.4}$$

and

$$C_n^0(v) = \frac{1}{2^{n+1}} \max_{u \in [0, 1]} \left(|E_{n+2}(u)| + |E_{n+1}(u)| \frac{|E_{n+2}(v)|}{|E_{n+1}(v)|} \right). \tag{4.5}$$

Proof. Equations (4.2) and (4.3) are direct consequences of the proof of the lemma 6. To obtain (4.5) we first observe that the righthand side of (4.3) is a continuous function of $u \in [0, 1]$. It remains to observe that the maximum is at a value u such that

$$\operatorname{sign}(E_{n+2}(u)) = -\operatorname{sign}\left(E_{n+1}(u) \frac{E_{n+2}(v)}{E_{n+1}(v)} \right).$$

This fact comes from the properties of the Euler splines. Finally (4.4) is a consequence of (4.2) and (4.5). ∎

The next result indicates the best choice of v for the interpolating problem with equispaced data on a uniform partition.

THEOREM 2. *We have*

$$\min_{0 \leq v \leq 1} C_n^0(v) = C_n^0(\tau_n^*)$$

and

$$C_n^0(\tau_n^*) = \frac{|E_{n+2}(\tau_{n+1}^*)|}{2^{n+1}} = \frac{|p_{n+1}(\tau_{n+1}^*, -1)|}{2^{n+1}}. \tag{4.6}$$

Proof. From (4.5) we have

$$C_n^0(v) \geqslant \frac{1}{2^{n+1}} \max_{u \in [0,1]} |E_{n+2}(u)| = \frac{|E_{n+2}(\tau_{n+1}^*)|}{2^{n+1}} \tag{4.7}$$

for all $v \in [0, 1]$ such that $p_n(v, -1) \neq 0$. But the righthand side of (4.7) is exactly the value of $C_n^0(\tau_n^*)$ because $\tau_n^* = \tau_{n+1}$ and $E_{n+2}(\tau_{n+1}) = 0$. ∎

Results similar to (4.6) appear elsewhere, for example [9, Theorem 5, p. 291], [2, theorem 3, p. 47], and [14].

EXAMPLE 1. If $\bar{E}_n$ is the Euler polynomial of degree n, we have

$$p_n(t, -1) = \frac{(-2)^n}{n!} \bar{E}_n(t)$$

(see [1, pp. 804–805]). Also the nth Euler number is given by $\bar{E}_n = 2^n \bar{E}_n(\frac{1}{2})$ and $\bar{E}_n(0) = -2((2^{n+1} - 1)/(n+1)) \bar{B}_{n+1}$ where $\bar{B}_n$ is the nth Bernoulli number. It follows that

(a) for n odd, $\tau_n^* = 0$ (or 1), $\tau_{n+1}^* = \frac{1}{2}$ and

$$C_n^0(0) = \frac{|\bar{E}_{n+1}|}{2^{n+1}(n+1)!},$$

(b) for n even, $\tau_n^* = \frac{1}{2}$, $\tau_{n+1}^* = 0$ (or 1) and

$$C_n^0\left(\frac{1}{2}\right) = 2\,\frac{2^{n+2} - 1}{(n+2)!}\,|\bar{B}_{n+2}|.$$

Remark 1. The results of this section can be applied directly to periodic functions on an interval $[a, b]$ when $b - a = Nh$ and N is even. For N odd we can prove the following asymptotic result based on the absolute convergence of the Laurent series $p_n(v, z)^{-1} = \sum_{j=-\infty}^{\infty} \alpha_j(v)\, z^j$.

THEOREM 3. *Let $v \in [0, 1]$ be such that $p_n(v, -1) \neq 0$. For any $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that for each N odd $\geqslant N(\varepsilon)$ there exists a periodic function $f \in \mathscr{C}$ of period $b-a$ such that*

$$C_n^0(u, v) - \varepsilon \leqslant \frac{\sup_{l \in \mathbb{Z}} |e_{l+u}|}{h^{n+1} \|f^{(n+1)}\|_\infty} \leqslant C_n^0(u, v). \quad \blacksquare$$

5. THE CASE $k > 0$

In this section we obtain bounds for the constants $C_n^k(u, v)$ and $C_n^k(v)$. These results are based on the following lemma.

LEMMA 7. *For $k = 1, ..., n$ we have*

$$\int_0^{n+1} \frac{|K_n^k(u, v, \theta)|}{n!} d\theta \leqslant \int_{\mathbb{R}} \int_0^1 \frac{|K_{n-k}^0(u, \mu, \theta)|}{(n-k)!} d\mu \, d\theta.$$

Proof. From lemma 5 and lemma 4 we have

$$\begin{aligned}
&\int_0^{n+1} \frac{|K_n^k(u, v, \theta)|}{n!} d\theta \\
&\quad \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} Q_{k-1}(v-\mu) \frac{|K_{n-k}^0(u, \mu, \theta - k)|}{(n-k)!} d\mu \, d\theta \\
&\quad = \sum_{i=-\infty}^{\infty} \int_{\mathbb{R}} \int_0^1 Q_{k-1}(v-i-\mu) \frac{|K_{n-k}^0(u, \mu, \theta - k - i)|}{(n-k)!} d\mu \, d\theta \\
&\quad = \sum_{i=-\infty}^{\infty} \int_0^1 \int_{\mathbb{R}} Q_{k-1}(v-i-\mu) \frac{|K_{n-k}^0(u, \mu, \theta)|}{(n-k)!} d\theta \, d\mu \\
&\quad = \int_{\mathbb{R}} \int_0^1 \sum_{i=-\infty}^{\infty} Q_{k-1}(v-i-\mu) \frac{|K_{n-k}^0(u, \mu, \theta)|}{(n-k)!} d\theta \, d\mu
\end{aligned}$$

and the result follows because $\sum_{i=-\infty}^{\infty} Q_{k-1}(v-i-\mu) = 1$. $\blacksquare$

By a similar method we can prove the following result.

LEMMA 8. *Let $v \in [0, 1]$ be such that $p_n(v, -1) \neq 0$. If $f \in AC_{\text{loc}}^{n+1}(\mathbb{R})$ and f is of polynomial growth, then*

$$p_n(v, E) \, e_{l+u}^{(k)} = h^{n-k+1} \int_{\mathbb{R}} \int_0^1 \frac{K_{n-k}^0(u, \mu, \theta)}{(n-k)!} E p_{k-1}(v-\mu, E) \, f_{l+\theta}^{(n+1)} \, d\mu \, d\theta$$

for $k = 1, ..., n$.

A direct consequence of lemma 6 and the properties of the Euler splines is the following result.

LEMMA 9.

$$\int_{\mathbb{R}} \int_0^1 \frac{|K_n^0(u, \mu, \theta)|}{n!} \, d\mu \, d\theta = \frac{E_{n+2}^2(u) - E_{n+1}(u) \, E_{n+3}(u)}{2^{n+1}}. \quad ∎$$

THEOREM 4. *Let $v \in [0, 1]$ be such that $p_n(v, -1) \neq 0$. If we set*

$$D_n^k(u, v) = \frac{1}{|p_n(v, -1)|} \int_{\mathbb{R}} \int_0^1 \frac{|K_{n-k}^0(u, \mu, \theta)|}{(n-k)!} \, d\mu \, d\theta$$

then

$$C_n^k(u, v) \leqslant D_n^k(u, v) \tag{5.1}$$

and

$$D_n^k(u, v) = \frac{1}{2^{n-k+1} \, |E_{n+1}(v)|} \left(E_{n-k+2}^2(u) - E_{n-k+1}(u) \, E_{n-k+3}(u)\right) \tag{5.2}$$

for $k = 1, ..., n$.

Proof. Equation (5.1) is a direct consequence of lemma 7 and (5.2) comes from lemma 9. ∎

LEMMA 10. *For any $u \in [0, 1]$ and $n \geqslant 0$ we have*

$$\left(\frac{2}{\pi}\right)^{2n+2} \leqslant E_{n+2}^2(u) - E_{n+1}(u) \, E_{n+3}(u) \leqslant \left(\frac{2}{\pi}\right)^{2n}. \tag{5.3}$$

Proof. We consider two cases. For $n = 0$, $E_1(u) = 1$, $E_2(u) = 2u - 1$, and $E_3(u) = 2u(u-1)$. Then

$$E_2^2(u) - E_1(u) \, E_3(u) = 1 + 2u(u-1)$$

and the result follows for $n = 0$. For $n > 0$, we first observe that

$$E_{n+2}^2(u) - E_{n+1}(u) \, E_{n+3}(u) = E_{n+2}^2(u) + |E_{n+1}(u)| \, |E_{n+3}(u)|.$$

We obtain the result using (2.5). ∎

If we set

$$D_n^k(v) = \sup_{0 \leqslant u \leqslant 1} D_n^k(u, v)$$

then, from (5.2) and (5.3)

$$C_n^k(u, v) \leqslant D_n^k(u, v) \leqslant \frac{(\pi^2/2)^{k-n}}{2\,|p_n(v, -1)|}.$$

Finally

$$D_n^k(\tau_n^*) \leqslant \frac{(\pi^2/2)^{k-n}}{2\,|p_n(\tau_n^*, -1)|}. \tag{5.4}$$

Remark 2. We can show that

$$\max_{u \in [0,1]} (E_{n+2}^2(u) - E_{n+1}(u)\, E_{n+3}(u)) = E_{n+2}^2(1) - E_{n+1}(1)\, E_{n+3}(1),$$

for $0 \leqslant n \leqslant 3$, but it is an open problem for $n > 3$. From this result we can obtain Table I.

EXAMPLE 2. As in section 4, we have

(a) for n odd, $\tau_n^* = 0$ (or 1), $p_n(\tau_n^*, -1) = (-2)^{n+1}((2^{n+1}-1)/(n+1)!)\,\bar{B}_{n+1}$, and

$$C_n^k(0) \leqslant D_n^k(0) \leqslant \frac{(\pi^2/2)^{k-n}(n+1)!}{2^{n+2}(2^{n+1}-1)\,|\bar{B}_{n+1}|},$$

(b) for n even, $\tau_n^* = \frac{1}{2}$, $p_n(\tau_n^*, -1) = \bar{E}_n/n!$, and

$$C_n^k\left(\frac{1}{2}\right) \leqslant D_n^k\left(\frac{1}{2}\right) \leqslant \frac{(\pi^2/2)^{k-n}\, n!}{2\,|\bar{E}_n|}.$$

TABLE I

Some Values of $D_n^k(\tau_n^*)$

$k \mid n$	2	3	4	5
1	1/6	1/24	1/75	···
2	1	1/4	3/45	1/48
3		3/2	2/5	15/144
4			12/5	15/24
5				15/4

6. Conclusion

We have obtained expressions for the optimal bounds when $k=0$. For $k>0$, we have obtained closed expressions to bound the constants $C_n^k(u, v)$ and $C_n^k(v)$. If we put together (4.6) and (5.4), and use (2.5) and (2.6) we obtain the following result.

Theorem 5. *Let $v=\tau_n^*$. If $f\in\mathscr{C}$, s is the n-degree interpolating spline of f such that $f_{l+v}=s_{l+v}$ for all $l\in\mathbb{Z}$, and $e=f-s$, then*

$$\|e^{(k)}\|_\infty \leqslant \frac{(\pi^2/2)^k}{2\pi^n} h^{n+1-k} \|f^{(n+1)}\|_\infty$$

for $k=0, ..., n$.

A similar result has already been presented in [9, Theorem 6, p. 293] for $k=0$ and n odd.

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